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# ON THE CONVERGENCE AND OVERCONVERGENCE OF MITTAG-LEFFLER SERIES

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Abstract. We consider series defined by means of the Mittag-Leffler functions and their Prabhakar generalizations and study the behaviour of such series on the peripheries of their convergence domains. Analogues of the classical theorems for the power series like overconvergence, as well as Hadamard type theorems are proposed.

## 1. Introduction

The special functions, defined in the whole complex plane $\mathbb{C}$ by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{1.1}$$

with $\alpha,\ \beta \in \mathbb{C},\quad Re(\alpha) > 0$, are known as Mittag-Leffler (M-L) functions ([1], Section 18.1). The first was introduced by Mittag-Leffler (1902-1905) who investigated some of its properties, while the other first appeared in a paper of Wiman (1905). Prabhakar [12] generalizes (1.1) by introducing the function $E^{\gamma}_{\alpha,\,\beta}$ of the form

$$E^{\gamma}_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \tag{1.2}$$

where $(\gamma)_k$ is the Pochhammer symbol ([1], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1).$$

For $\gamma = 1$ this function coincides with $E_{\alpha,\,\beta}$, while for $\gamma = \beta = 1$ with $E_\alpha$.

In the previous papers [9, 10], the author considered series in systems of Mittag-Leffler type functions and, resp. in [11], series in the multi-index ($2m$-indices) analogues

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of the M-L functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus [4]. Their convergence in the complex plane $\mathbb{C}$ is studied and Cauchy-Hadamard, Abel, Tauberian and Fatou type theorems are proved. In this paper the overconvergence of series in Mittag-Leffler functions and their three-parametric Prabhakar generalizations are also studied. Such a kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions (as for example, in Kiryakova [3] and Sandev, Tomovski and Dubbeldam [13]).

## 2. Previous results

Consider now the first of the functions (1.1) for positive indices $\alpha = n \in \mathbb{N}$ and also generalized Mittag-Leffler functions (1.2) for indices of the kind $\beta = n$; $n = 0, 1, 2, \dots$ , namely:

$$E_n(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(nk+1)}, \ n \in \mathbb{N};$$

$$E^{\gamma}_{\alpha,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!}, \qquad \alpha, \ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \ n \in \mathbb{N}_0. \tag{2.1}$$

In this section we give some results related to the asymptotic formula for "large" values of indices of the functions (2.1) that can be seen e.g. in [9], [10]. Furthermore we need them to prove the main theorems.

**Remark 2.1.** *Given a number $\gamma$, suppose that some coefficients in* (2.1) *are equal to zero, that is, there exists a number $p \in \mathbb{N}_0$, such that the representation* (2.1) *can be written as follows:*

$$E^{\gamma}_{\alpha,n}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}.$$

More precisely, as it is given in [10], if $\gamma$ is different from zero, then $p = 0$ for each positive integer $n$ and $p = 1$ for $n = 0$.

Further, asymptotic formulae for "large" values of the indices $n$ are given for $z, \alpha, \gamma \in \mathbb{C}, \gamma \neq 0, Re(\alpha) > 0$. Namely, there exist entire functions $\theta_n$ and $\theta^{\gamma}_{\alpha,n}$ such that the functions (2.1), have the following asymptotic formulae:

$$E_n(z) = 1 + \theta_n(z) \quad (n \in \mathbb{N}), \quad \text{and} \quad \theta_n(z) \to 0 \quad \text{as } n \to \infty;$$

$$E^{\gamma}_{\alpha,n}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p + n)} z^p \left(1 + \theta^{\gamma}_{\alpha,n}(z)\right) \quad (n \in \mathbb{N}_0), \quad \text{and} \quad \theta^{\gamma}_{\alpha,n}(z) \to 0 \quad \text{as } n \to \infty,$$

with the corresponding $p$, depending on $\gamma$. Moreover, on the compact subsets of the complex plane $\mathbb{C}$, the convergence is uniform and

$$\theta_n(z) = O\left(\frac{1}{n!}\right), \quad \theta^{\gamma}_{\alpha,n}(z) = O\left(\frac{1}{n^{Re(\alpha)}}\right) \quad (n \in \mathbb{N}). \tag{2.2}$$

**Remark 2.2.** *If $\gamma = 0$, the functions* (2.1) *take the simplest form $E^{0}_{\alpha,n}(z) = \dfrac{1}{\Gamma(n)}$ for $n \in \mathbb{N}$, and $E^{0}_{\alpha,n}(z) = 0$ for $n = 0$.*

Now, let us specify the families of Mittag-Leffler type functions

$$\left\{\widetilde{E}_n(z)\right\}_{n=0}^{\infty}, \quad \left\{\widetilde{E}^{\gamma}_{\alpha,n}(z)\right\}_{n=0}^{\infty}; \qquad \alpha,\gamma\in\mathbb{C},\ Re(\alpha)>0,$$

as follows below, namely:

$$\widetilde{E}_0(z)=1,\ \widetilde{E}_n(z)=z^n\,E_n(z),\quad \widetilde{E}^0_{\alpha,0}(z)=0,\ \widetilde{E}^0_{\alpha,n}(z)=\Gamma(n)\,z^n\,E^0_{\alpha,n}(z);\ \ n\in\mathbb{N},$$

$$\widetilde{E}^{\gamma}_{\alpha,n}(z)=\frac{\Gamma(\alpha p+n)}{(\gamma)_p}\,z^{n-p}\,E^{\gamma}_{\alpha,n}(z),\ \ \gamma\neq 0,\ \ n\in\mathbb{N}_0,$$

and let us consider series in these functions of the form:

$$\sum_{n=0}^{\infty} a_n\widetilde{E}_n(z),\qquad \sum_{n=0}^{\infty} a_n\widetilde{E}^{\gamma}_{\alpha,n}(z), \tag{2.3}$$

with complex coefficients $a_n$ $(n=0,1,2,...)$.

Our objective is to study the convergence of the series (2.3) in the complex plane and to propose theorems, corresponding to the classical results for the power series. Beginning with the domain of convergence of the series (2.3), we recall [9, 10] that it is the open disk $D(0;R)=\{z:\ |z|<R, z\in\mathbb{C}\}$ with a radius of convergence

$$R=\left[\limsup_{n\to\infty}(\ |a_n|\ )^{1/n}\right]^{-1}. \tag{2.4}$$

More precisely, both series are absolutely convergent in the disk $D(0;R)$ and they are divergent in the domain $|z|>R$. The cases $R=0$ and $R=\infty$ fall in the general case. Farther, analogously to the classical Abel lemma, if any of the series (2.3) converges at the point $z_0\neq 0$, then it is absolutely convergent in the disk $D(0;|z_0|)$. Moreover, inside the disk $D(0;R)$, i.e., on each closed disk $|z|\le r<R$ ($R$ defined by (2.4)), the series is uniformly convergent. Another interesting result is the Abel type theorem which refers to the existence of the limit of the series sums at the point $z_0$ from the boundary $\partial D(0;R)=C(0;R)$, when $z$ belongs to a suitable angular domain with a vertex at the point $z=z_0$. Namely, the limit of the sum of these series, are equal to the corresponding series sum at the point $z_0$. A result, giving relation between the convergence (divergence) of the series (2.3) at points on the boundary of its disk of convergence and the regularity (singularity) of its sum at such points is formulated below. Analogical propositions have been established also for series in the Laguerre and Hermite polynomilas by Rusev, as well as in Mittag-Leffler type systems (see e.g. [11]). Here we give such a type of theorem for the Mittag-Leffler systems (for the line of proof, see [11]) as follows.

**Theorem 2.1** (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions $\lim_{n\to\infty} a_n=0$, $\limsup_{n\to\infty}(\ |a_n|\ )^{1/n}=1$, and $F(z)$ be the sum of any of the series* (2.3) *in the unit disk $D(0;1)$. Let $\sigma$ be an arbitrary arc of the unit circle $C(0;1)$ with all its points* (*including the ends*) *regular to the function $F$. Then the series* (2.3) *converges, even uniformly, on the arc $\sigma$.*

## 3. Overconvergence theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup_{n\to\infty}(|a_n|)^{1/n}=R^{-1}$, $0<R<\infty$ and $f(z)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n z^n$ in the open disk $D(0;R)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0; R). \tag{3.1}$$

In order to introduce the next two definitions ([6, Vol. 2, p. 500]) and to expose the results, obtained in this section, we first set

$$s_p(z) = \sum_{k=0}^{p} a_k z^k, \quad S_p(z) = \sum_{k=0}^{p} a_k \; \widetilde{E}_k(z), \text{ or } \quad \text{resp. } S_p(z) = \sum_{k=0}^{p} a_k \; \widetilde{E}^{\gamma}_{\alpha,\, k}(z),$$

for all the values $p = 0, 1, 2, \dots$.

**Definition 3.1.** *A power series with a finite radius of convergence $0 < R < \infty$ is said to be overconvergent, if there exist a subsequence $\{s_{p_k}\}_{k=0}^{\infty}$ of the partial-sums sequence $\{s_p\}_{p=0}^{\infty}$ and a region $G$, containing the open disk $D(0; R)$ as a regular part ($G \cap \partial D(0; R) \neq \varnothing$), such that $\{s_{p_k}\}$ uniformly converges inside $G$. We say that the function $f$ (or the series* (3.1)*), possesses Hadamard gaps, if there exist two sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$, having the properties $q_{n-1} \leq p_n \leq q_n/(1+\theta)$ ($\theta > 0$) and $a_k = 0$ for $p_n < k < q_n$ ($n = 0, 1, 2, \dots$).*

**Remark 3.1.** *To introduce the corresponding notions 'overconvergence' and 'gaps' for the series* (2.3)*, the expression $z^n$ has to be replaced by $\widetilde{E}_n(z)$, respectively $\widetilde{E}^{\gamma}_{\alpha,n}(z)$, and the sequence $\{s_{p_k}\}$ by the corresponding sequence $\{S_{p_k}\}$.*

Thus, beginning with the domain of convergence and series behaviour near its boundary, passing through the possible uniform convergence on an arbitrary closed arc of the boundary, we come to the natural question: "*What type of conditions should be imposed on the power series that ensure the existence of subsequence $\{s_{p_k}\}$, convergent outside the disk of convergence?*". The answer to this question is given in the early 20th century by Ostrowski [7], [8], see also [5]. Namely, his classical result states that a given power series with Hadamard gaps and existing regular points on the boundary of convergence disk is overconvergent. We draw the attention to the fact that merely the existence of Hadamard gaps does not imply overconvergence. For example, the power series $\sum\limits_{n=0}^{\infty} a_{k_n} z^{k_n}$ with $k_{n+1} \geq (1+\theta)k_n$ ($\theta > 0$) and $\limsup\limits_{n\to\infty} (\,|a_{k_n}|\,)^{1/k_n} = 1$ possesses Hadamard gaps but nevertheless it is not overconvergent. Its natural boundary of analyticity is the unit circle $|z| = 1$ and that is nothing but the theorem about the gaps, belonging to Hadamard [2].

**Theorem 3.1** (of overconvergence)**.** *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup\limits_{n\to\infty} (\,|a_n|\,)^{1/n} = 1$, $F(z)$ be the sum of the series* (2.3) *on the unit disk $D(0; 1)$, $F(z)$ have at least one regular point, belonging to the circle $C(0; 1)$, and let $F(z)$ possesse Hadamard gaps. Then the series* (2.3) *is overconvergent.*

*Proof.* Here we expose the proof for the second of the series (2.3) and we only note that the other goes analogously. Without loss the generality we suppose that the point $z_0 = 1$ is regular to the function $F$. That means that $F$ is analytically continuable in a neighbourhood $U$ of the point 1. Denoting $\widetilde{U} = U \cup D(0;1)$, we define the function $\psi$ in the region $\widetilde{U}$ by the equality

$$\psi(z) = F(z), \quad z \in D(0;1),$$

i.e. $\psi$ is a single valued analytical continuation of $F$ in the domain $\widetilde{U}$.

Letting $\theta > 0$ and taking $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ with the properties $q_n \geq (1+\theta)p_n$ and $a_k = 0$ for $p_n < k < q_n$ $(n = 0, 1, 2, \dots)$, we define the auxiliary function

$$\varphi_n(z) = \psi(z) - S_{p_n} = \psi(z) - \sum_{k=0}^{p_n} a_k \widetilde{E}_{\alpha,\,k}^{\gamma}(z)(z). \tag{3.2}$$

In order to prove that the sequence $\{S_{p_n}\}$ is uniformly convergent inside the region $\widetilde{U}$, we are going to apply the Hadamard theorem for the three disks [6, Vol. 2]. To this end, taking $0 < \delta < 1/2$ and $0 < \omega < \delta$, we consider the three circles $C_1, C_2, C_3$, centered at the point $1/2$ and having respectively radii $1/2 - \delta$, $1/2 + \omega$, $1/2 + \delta$, such that $C_3 \subset \widetilde{U}$ and after that set

$$M_{n,j} = \max_{z \in C_j} |\varphi_n(z)| \quad j = 1, 2, 3; \qquad M = \max_{z \in C_3} |\psi(z)|.$$

Before evaluating $|\varphi_n(z)|$ we come back to (2.2). Just mention that since $\lim\limits_{n\to\infty} n^{-Re(\alpha)} = 0$, there exist number $B$ such that $|1 + \theta_n(z)| \leq B$ for all the values of $n \in \mathbb{N}$ on an arbitrary compact subset of $\mathbb{C}$. Now, letting $0 < \eta < \delta/2$ implies the existence of $A = A(\eta)$ such that $|a_k| \leq AB^{-1}(1-\eta)^{-k}$. To find an upper estimation of $|\varphi_n(z)|$ we intend to consider three different cases.

**1.** First, let $z \in C_1 \subset D(0;1)$. In the unit disk, according to (3.2), we have

$$\varphi_n(z) = \sum_{k=q_n}^{\infty} a_k \widetilde{E}_{\alpha,\,n}^{\gamma}(z).$$

Therefore,

$$\begin{aligned}|\varphi_n(z)| &\leq \sum_{k=q_n}^{\infty} |a_n \widetilde{E}_{\alpha,\,n}^{\gamma}(z)| = \sum_{k=q_n}^{\infty} |a_k z^k (1 + \theta_k(z))| = \sum_{k=q_n}^{\infty} |a_k||1 + \theta_k(z)||z^k| \\ &\leq A \sum_{k=q_n}^{\infty} (1-\eta)^{-k}(1-\delta)^k = A\left(1 - \frac{1-\delta}{1-\eta}\right)^{-1} \left(\frac{1-\delta}{1-\eta}\right)^{q_n},\end{aligned}$$

whence

$$M_{n,1} = O\left(\left(\frac{1-\delta}{1-\eta}\right)^{q_n}\right) = O\left(\left(\frac{1-\delta}{1-\eta}\right)^{(1+\theta)p_n}\right). \tag{3.3}$$

**2.** Now, let $z \in C_3$. In this case

$$\begin{aligned}|\varphi_n(z)| &= |\psi(z) - S_{p_n}| = |\psi(z) - \sum_{k=0}^{p_n} a_k \widetilde{E}_{\alpha,\,k}^{\gamma}(z)| \leq |\psi(z)| + \sum_{k=0}^{p_n} |a_k \widetilde{E}_{\alpha,\,k}^{\gamma}(z)| \\ &\leq M + \sum_{k=0}^{p_n} |a_k||1 + \theta_k(z)||z^k| \leq M + A \sum_{k=0}^{p_n} \left(\frac{1+\delta}{1-\eta}\right)^k = O\left(\left(\frac{1+\delta}{1-\eta}\right)^{p_n}\right),\end{aligned}$$

and therefore

$$M_{n,3} = O\left(\left(\frac{1+\delta}{1-\eta}\right)^{p_n}\right). \tag{3.4}$$

**3.** Furthermore, let $z \in C_2$. Then, in view of (3.3) and (3.4) and according to the Hadamard theorem for the three disks (for details see [6, Vol. 2, formula (3.2:2)]), we can write

$$M_{n,2} = O\left(\left(\left(\frac{1-\delta}{1-\eta}\right)^{(1+\theta)\ln\frac{1+2\delta}{1+2\omega}}\left(\frac{1+\delta}{1-\eta}\right)^{\ln\frac{1+2\omega}{1-2\delta}}\right)^{p_n}\right). \tag{3.5}$$

Note that the limit of the inner part of (3.5) is equal to

$$a = (1-\delta)^{(1+\theta)\ln(1+2\delta)}(1+\delta)^{-\ln(1-2\delta)}$$

when $\omega$ and $\eta$ tend to 0. Moreover, if $\delta$ tends to 0 then $a < 1$. Indeed, taking the logarithm of $a$, we have

$$\begin{aligned}\ln a &= (1+\theta)\ln(1+2\delta)\ln(1-\delta) - \ln(1-2\delta)\ln(1+\delta)\\ &= (1+\theta)(2\delta+O(\delta^2))(-\delta+O(\delta^2)) - (-2\delta+O(\delta^2))(\delta+O(\delta^2)) = -2\theta\delta^2+O(\delta^3).\end{aligned}$$

Therefore $\ln a < 0$ when $\delta \to 0$ and for this reason $a < 1$ if $\delta$ tends to 0. That is why, $\lim\limits_{n\to\infty} M_{n,2} = 0$, whence $\{S_{p_n}\}$ is uniformly convergent inside the region $\widetilde{U}$. □

**Theorem 3.2** (of Hadamard about the gaps). *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence of complex numbers satisfying the condition* $\limsup\limits_{n\to\infty}(\,|a_{k_n}|\,)^{1/k_n} = 1$, $k_{n+1} \geq (1+\theta)k_n$ $(\theta > 0)$, $a_k = 0$ *for* $k_n < k < k_{n+1}$ *and* $F(z)$ *be the sum of any of the series* (2.3) *in the unit disk* $D(0;1)$. *Then all the points of the unit circle* $C(0;1)$ *are singular for the function* $F$, *i.e. the unit circle is a natural boundary of analyticity for the corresponding series.*

*Proof.* Let $|z_0| = 1$ and $z_0$ be regular for $F$, $p_n = k_n$, $q_n = k_{n+1}$. Therefore, in accordance with Theorem 3.1, $S_n$ uniformly converges in a neighbourhood of $z_0$. But the radius of convergence is $R = 1$ and we come to contradiction. □

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